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THE CONTROL OF A MECHANICAL SYSTEM WITH UNKNOWN PARAMETERS BY A BOUNDED FORCE†

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A mechanical system whose dynamics can be described by Lagrange's equations of the second kind is considered. It is assumed that the kinetic energy matrix of the system is unknown and the system is subject to uncontrollable bounded external forces (such a situation occurs, for example, if the load carried by a manipulator remains unknown). A control law is constructed which enables the system to be transferred from an arbitrary initial state to a given final state in a finite time using a force of bounded modulus. In the algorithm proposed linear feedback is used with piecewise-constant coefficients: the coefficients increase as the system approaches the final state. The algorithm rests on the second Lyapunov method. The results of a numerical model of the dynamics of a double-link unit are presented. © 1997 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

We consider a mechanical system whose dynamics can be described by Lagrange's equations of the second kind. We denote by $q \in R^n$ the vector formed by the generalized coordinates of the system, by $p = \dot{q}$ the vector of generalized velocities, by S the external forces acting on the systems, by u the control forces, and by $T(q,p) = \langle A(q)p, p\rangle/2$ the kinetic energy of the system. Here $A(q) \in C^1$ is a positive definite symmetric matrix and $\langle \cdot, \cdot \rangle$ denotes the scalar product.

The equations of motion of the system in Lagrange form are

$$\frac{d}{dt}\frac{\partial T}{\partial p} - \frac{\partial T}{\partial q} = S + u \tag{1.1}$$

We impose the conditions

$$|S| \leqslant S_0, \; S_0 > 0 \tag{1.2}$$

$$|u| \leqslant U, \; U > 0 \tag{1.3}$$

on the n-dimensional vectors of external forces S and control forces u. Here and everywhere below $|\cdot|$ denotes the Euclidean norm of a vector or matrix (by the norm of a matrix we mean the norm of the corresponding operator in Euclidean space).

The forces S will be assumed to be unknown and treated as external perturbations. Along with them, other forces of specified magnitude may act on the system. However, we assume that the control resources are large enough to compensate for these specified forces, U being the maximum admissible control intensity remaining after such compensation.

We shall assume that $A(q)$ is an unknown matrix whose eigenvalues lie in the range $[m^2, M^2]$, $0 < m \leqslant M$ for any q and the partial derivatives of $A(q)$ are uniformly bounded in the norm, i.e.

$$\forall z \in R^n \quad m^2 z^2 \leqslant \langle A(q)z, z\rangle \leqslant M^2 z^2$$
$$|\partial A(q)/\partial q_i| \leqslant D, \quad D > 0, \quad i = 1, \ldots, n \tag{1.4}$$

It is required to construct a control function that satisfies (1.3) and takes the system from an arbitrary initial state (q_*, p_*) to a prescribed final state $(\bar{q}, 0)$ in a finite time.

An illustrative example of this formulation of the problem is given by the problem of controlling a system of connected rigid bodies whose precise mass-inertial characteristics are unknown. In this case

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not only the inertia matrix of the system, but also the forces acting on the bodies remain unknown. Apart from these forces, the system may be subject to other external perturbations. The problem of transporting a load of unknown mass by a manipulator is a special case of this problem.

The formulation of problems on the construction of bounded control in a mechanical system under conditions of uncertainty considered in [1–4] is closest to the above. Thus, a control function was constructed in [1] using a proportional-integro-differential (PID) controller which transferred system (1.1) with unknown matrix $A(q)$ to a prescribed equilibrium state in an infinite time when there are no external forces. In [2, 3] an approach based on decomposition was developed, which enables one to control a system with known inertia matrix subject to uncontrollable external perturbations using a bounded force. A control law, which makes use of linear feedback with piecewise-constant coefficients, has been proposed in [4] for controlling (1.1) with $S = 0$. This approach is considered below in the case of non-zero external perturbations.

We shall seek a control function as a linear feedback in terms of the generalized coordinates and velocities $u = -ap - b(q - \bar{q})$, where a and b are piecewise-constant functions. We know [1] that if a and b are arbitrary positive constants and $S = 0$, the equilibrium $(\bar{q}, 0)$ of the closed system (1.1) is asymptotically stable as a whole, i.e. the control in question transfers the system from an arbitrary initial state to the final state $(\bar{q}, 0)$ in an infinite time. This feedback control law with constant coefficients has a significant disadvantage, namely, the control forces do not satisfy (1.3) far from the terminal state and they are small near this state. The control resources given by (1.3) are not completely utilized, as a result of which the time of motion is infinite. To accelerate the transition of the system we shall vary a and b as the final state is approached.

Without loss of generality we can assume that $\bar{q} = 0$ (otherwise it suffices to take $q - \bar{q}$ as the vector of generalized coordinates). We will reformulate the original problem as follows. Suppose that the initial conditions $q(0) = q_*$, $p(0) = p_*$ and constants m, M, D, U are given. It is required to specify how the feedback factors a and b in the control function

$$u = -ap - bq \tag{1.5}$$

should be varied so that for any external perturbations S that satisfy (1.2) system (1.1), (1.5) arrives at the position (0, 0) in a finite time and restrictions (1.3) on u are satisfied along the trajectory.

2. DESCRIPTION OF THE CONTROL ALGORITHM

Consider the function

$$W(q,p) = M^2p^2 + (M^4p^4 + U^2q^2/2)^{1/2} \tag{2.1}$$

The quantity $W(q,p)$ has the dimension of energy and characterizes the distance between (q,p) and the final state (0, 0). The level set $W(q,p) = C$ of W in phase space is an ellipsoid $4CM^2p^2 + U^2q^2 = 2C^2$, which collapses to the origin (0, 0) of the system of coordinates as $C \to 0$. We put

$$D_1 = \sqrt{n}D/2, \quad W_0 = M^2U/(2\sqrt{2}D_1), \quad W_k = W_0/2^k \tag{2.2}$$

and define a family of ellipsoids $W(q,p) = W_k$, where k runs through the set of integers. Suppose that the point (q_*, p_*) corresponding to the phase state of the original system at the initial time $t = 0$ lies on the ellipsoid $W(q,p) = W_k$ or inside it, but not outside the ellipsoid $W(q,p) = W_{k^*+1}$, i.e. $W_{k^*+1} < W(q_*, p_*) \le W_{k^*}$. We denote by t_{k^*+1} the first time when the trajectory of the system reaches the ellipsoid $W(q,p) = W_{k^*+1}$. Below it will be shown that for the chosen control algorithm the trajectory of the system tends to the origin, which means that such a moment exists. We put $q(t_{k^*+1}) = q_{k^*+1}$, $p(t_{k^*+1}) = p_{k^*+1}$ and denote by t_{k^*+2} the first time when the trajectory of the system reaches the ellipsoid $W(q,p) = W_{k^*+2}$. We put $q(t_{k^*+2}) = q_{k^*+2}$, $p(t_{k^*+2}) = p_{k^*+2}$ and so on.

The sequence $\{t_k\}$, $k = k_* + 1, k_* + 2, \ldots$ defines the instants when the feedback factors in (1.5) vary. We define the values of these factors in the time interval $[t_k, t_{k+1})$ as follows:

$$b_k = U^2/(4W_k), \quad a_k^2 = m^2b_k \tag{2.3}$$

The initial values of the factors are determined from (2.3) when $k = k_*$. In the phase space q, p the trajectory of motion of the mechanical system under consideration will therefore consist of segments of trajectories of different systems of differential equations: the kth interval connects (q_k, p_k) and (q_{k+1}, p_{k+1}) and corresponds to a system of the form (1.1), (1.5) in which the gains $a = a_k$, $b = b_k$ are constant and given by (2.3). All points (q_k, p_k) lie on the corresponding ellipsoids $W(q, p) = W_k$, $k > k_*$.

Remark. The trajectory of the system tends to the origin (0, 0) of the system of coordinates, but W is not, in general, a decreasing function along the trajectory. Thus, along with the points (q_k, p_k), the trajectory can also have other points of intersection with the ellipsoids $W(q, p) = W_k$. Suppose, for example, that once the new factors are assigned at time t_k, the trajectory of the system begins to "move away" from the final position (0, 0) and it intersects the ellipsoid with number $k - 1$ again at some $t' > t_k$. At the instant t' the index k and the gains a_k and b_k do not vary. They take new values only when the trajectory reaches the ellipsoid $W(q, p) = W_{k+1}$: k increases by one, a increases by a factor of $\sqrt{2}$ and b by a factor of two. Therefore the gains a_k and b_k in (1.5) depend on the known parameters m, M, U, D of the problem and on the current value of k. At each instant the index k is equal to the number of the minimum ellipsoid that has already been visited by the trajectory of the system.

3. JUSTIFICATION OF THE ALGORITHM

We shall study the behaviour of the trajectory of the kth system for some $k > k_*$. The section of the trajectory that is of interest to us starts at (q_k, p_k) at time t_k and ends in accordance with the algorithm at time t_{k+1} on the ellipsoid $W(q, p) = W_{k+1}$. Since the existence of an intersection of the trajectory with the $(k + 1)$st ellipsoid has not been shown so far, we shall assume that $t_{k+1} = \infty$ if there is no such intersection. Below it will be shown that $t_{k+1} < \infty$.

We introduce the Lyapunov function

$$V^k(q,p) = T(q,p) + b_k q^2/2 + \varepsilon_k \langle A(q)q, p\rangle \tag{3.1}$$

The number $\varepsilon_k > 0$ is to be determined. The expression for $V^k(q, p)$ contains the inertia matrix $A(q)$, which is assumed to be unknown. We shall estimate the value of this function at the point (q, p) in phase space by means of known quantities. Suppose that ε_k satisfies the condition

$$\varepsilon_k^2 < m^2 b_k/(4M^4) \tag{3.2}$$

We find a lower bound of $V^k(q, p)$ using (1.4) and the Cauchy inequality $\varepsilon_k M^2 |q||p| \leqslant \varepsilon_k^2 M^4 q^2/m^2 + m^2p^2/4$ as follows:

$$V^k(q,p) \geqslant \frac{b_k q^2}{2} + \frac{m^2 p^2}{2} - \varepsilon_k M^2 |q||p| \geqslant \left(\frac{b_k}{2} - \frac{\varepsilon_k^2 M^4}{m^2}\right) q^2 + \left(\frac{m^2}{4}\right) p^2$$

from which, taking (2.2) into account, we obtain the inequality

$$V_-^k(q,p) \leqslant V^k(q,p), \quad V_-^k(q,p) = (b_k q^2 + m^2 p^2)/4 \tag{3.3}$$

We now estimate $V^k(q, p)$ from above using (1.4) and the inequality $\varepsilon_k |q||p| \leqslant \varepsilon_k^2 q^2/2 + p^2/2$ as follows:

$$V^k(q,p) \leqslant \frac{b_k q^2}{2} + \frac{M^2 p^2}{2} + \varepsilon_k M^2 |q||p| \leqslant \frac{(b_k + \varepsilon_k^2 M^2)}{2} q^2 + M^2 p^2$$

Since $m^2/(4M^2) < 1$, (3.2) implies that $\varepsilon_k^2 M^2 < b_k$, whence we obtain the inequality

$$V^k(q,p) \leqslant V_+^k(q,p), \quad V_+^k(q,p) = b_k q^2 + M^2 p^2 \tag{3.4}$$

We shall establish relations between the quadratic forms $V_+^k(q, p)$ and the function $W(q, p)$, whose level sets generate the family of ellipsoids defined above. The equality

$$2V_+^k(q_k, p_k) = W_k \tag{3.5}$$

holds.

For the proof we substitute the expression for b_k from (2.3) into expression (3.4) for V_+^k. We obtain

$$V_+^k(q_k, p_k) = \frac{U^2 q_k^2 + 4W_k M^2 p_k^2}{4W_k} \tag{3.6}$$

By construction (q_k, p_k) lies on the ellipsoid with number k. By the definition (2.1) of W it follows that $W_k = W(q_k, p_k) = M^2 p_k^2 + (M^4 p_k^4 + U^2 q_k^2/2)^{1/2}$. Using this equality the numerator in (3.6) can be reduced to the form $2W_k^2$, which implies (3.5).

Equality (3.5) means that for any k the ellipsoid with number k is a level set of the quadratic form $V_+^k(q,p)$ corresponding to $W_k/2$. Suppose that the system is in a state (q,p) at time t, where $t_k < t < t_{k+1}$. According to the algorithm, (q,p) lies outside the $(k+1)$st ellipsoid, so $V_+^{k+1}(q,p) = b_{k+1}q^2 + M^2p^2 \geqslant W_{k+1}/2$. By (2.2) and (2.3) the equalities $W_{k+1} = W_k/2$, $b_{k+1} = 2b_k$ hold, so $V_+^{k+1}(q,p) = 2b_k q^2 + M^2 p^2 \geqslant W_k/4$. We estimate the value of $V_+^k(q,p)$ at time t by its value $V_+^k(q_k, p_k)$ at time t_k as follows:

$$V_+^k(q,p) \geqslant (2b_k q^2 + M^2 p^2)/2 \geqslant W_k/8 = V_+^k(q_k, p_k)/4 \tag{3.7}$$

We now proceed to find the derivative $\dot{V}_k$. We differentiate V^k using (1.1) and (1.5) to get

$$\dot{V}^k(q,p) = -\varepsilon_k b_k q^2 - \left\langle \left[a_k I - \varepsilon_k A(q) - \frac{\varepsilon_k}{2}\sum_{i=1}^n q_i \frac{\partial A}{\partial q_i} \right] p, p \right\rangle - \varepsilon_k a_k \langle q, p\rangle + \langle S, \varepsilon_k q + p\rangle \tag{3.8}$$

where I is the identity matrix. Let us estimate the various terms in (3.8). By (1.4)

$$|\varepsilon_k A(q)| \leqslant \varepsilon_k M^2, \qquad \left| \frac{\varepsilon_k}{2}\sum_{i=1}^n q_i \frac{\partial A}{\partial q_i} \right| \leqslant \varepsilon_k D_1 |q|, \qquad D_1 = \frac{\sqrt{n}D}{2} \tag{3.9}$$

From the Cauchy inequality it follows that

$$|\varepsilon_k a_k \langle q, p\rangle| \leqslant \varepsilon_k^2 a_k q^2 + a_k p^2/4 \tag{3.10}$$

Using (1.2), (3.2) and (3.7) we estimate the last term in (3.8) as follows:

$$|\langle S, \varepsilon_k q + p\rangle| \leqslant S_0 |\varepsilon_k q + p| \leqslant S_0[5\varepsilon_k^2 q^2 + 5p^2/4]^{1/2} \leqslant S_0[5(b_k m^2 q^2/M^4 + p^2)/4]^{1/2} \leqslant$$

$$\leqslant \frac{\sqrt{5}S_0}{2M}\sqrt{V_+^k(q,p)} = \frac{\sqrt{5}S_0 V_+^k(q,p)}{2M\sqrt{V_+^k(q,p)}} \leqslant \frac{\sqrt{10}S_0}{M\sqrt{W_k}}(b_k q^2 + M^2 p^2) \tag{3.11}$$

Substituting (3.9)–(3.11) into (3.8), we arrive at the limit

$$\dot{V}^k(q,p) \leqslant -\varepsilon_k \left(b_k - \varepsilon_k a_k - \frac{\sqrt{10}S_0 b_k}{\varepsilon_k M\sqrt{W_k}} \right) q^2 -$$

$$-\left(\frac{3a_k}{4} - \varepsilon_k M^2 - \varepsilon_k D_1 |q| - \frac{\sqrt{10}MS_0}{\sqrt{W_k}} \right) p^2 \tag{3.12}$$

We define the parameter ε_k by the formula

$$\varepsilon_k = \min\left\{ \frac{mU}{8M^2 W_k^{1/2}}, \frac{mU^2}{16\sqrt{2}D_1 W_k^{3/2}} \right\} \tag{3.13}$$

Lemma 1. Let the condition

$$S_0 \leq \min\left\{\frac{mU}{16\sqrt{10}M}, \frac{\varepsilon_k M W_k^{1/2}}{2\sqrt{10}}\right\} \tag{3.14}$$

be satisfied. Then, by virtue of system (1.1), (1.5), the derivative $\dot{V}^k$ at those points of the trajectory that lie in the region $G = \{(q,p): |q| < 2\sqrt{(2)}W_k/U\}$ satisfies the inequality

$$\dot{V}^k(q,p) \leq -\varepsilon_k b_k q^2/4 - a_k p^2/8 \tag{3.15}$$

Proof. Inequality (3.2) is a consequence of (3.13), so (3.3), (3.4), (3.11) and (3.12) are satisfied. Using (2.3) and (3.13), we obtain

$$\varepsilon_k a_k \leq \frac{m^2 U\sqrt{b_k}}{8M^2 W_k^{1/2}} = \frac{m^2 b_k}{4M^2} \leq \frac{b_k}{4}, \quad \varepsilon_k M^2 \leq \frac{mU}{8W_k^{1/2}} = \frac{m\sqrt{b_k}}{4M^2} = \frac{a_k}{4} \tag{3.16}$$

Relationship (3.14) and formulae (2.3) imply that

$$\frac{\sqrt{10}S_0 b_k}{\varepsilon_k M\sqrt{W_k}} \leq \frac{b_k}{2}, \quad \frac{\sqrt{10}MS_0}{\sqrt{W_k}} \leq \frac{mU}{16W_k^{1/2}} = \frac{a_k}{8} \tag{3.17}$$

From (3.13) and the definition of G it follows that

$$\varepsilon_k D_1|q|\frac{mU^2}{32\sqrt{2}W_k^{3/2}}|q| \leq \frac{mU}{16W_k^{1/2}} = \frac{a_k}{8} \tag{3.18}$$

Substituting (3.16)–(3.18) into (3.12), we obtain (3.15).

Lemma 2. Suppose that conditions (3.14) are satisfied. Then the part of the trajectory corresponding to the time interval $[t_{k,k+1})$ lies wholly in G.

Proof. We shall verify that the initial point (q_k, p_k) of the trajectory belongs to G. By construction (q_k, p_k) belongs to the ellipsoid with number k, i.e. $M^2p_k^2 + (M^4p_k^4 + U^2q_k^2/2)^{1/2}$. It follows that $q^2_k \leq 2W_k^2/U^2$, and hence $(q_k, p_k) \in G$.

Suppose that the assertion of the lemma does not hold and let t' be the first instant when the trajectory reaches the boundary of G such that $t' > t_k$. By Lemma 1, V^k is strictly decreasing in G along the solutions of system (1.1), (1.5). Whence, and by (3.4) and (3.5), we obtain

$$V^k(q(t'),p(t')) < V^k(q_k,p_k) \leq V_+^k(q_k,p_k) = W_k/2$$

On the other hand, $q^2(t') = 8W_k^2/U^2$ on the boundary of G. By (2.3) and (3.3) it follows that

$$V^k(q(t'),p(t')) \geq V_-^k(q(t'),p(t')) \geq b_k q^2(t')/4 = W_k/2$$

This contradiction completes the proof of the lemma.

By (3.3) Lyapunov's function (3.1) is positive definite and (3.15) implies that its derivative is negative and non-zero outside the $(k+1)$st ellipsoid. Hence we can conclude that there is an instant $t_{k+1} < \infty$ such that the trajectory reaches the ellipsoid with number $k+1$. We shall verify that the control forces satisfy (1.3) on the section of the trajectory corresponding to the half-interval of time $[t_k, t_{k+1})$. We shall find a bound of the norm of the vector u using (2.3) for a_k and inequality (3.3) as follows:

$$|u|^2 = |b_k q + a_k p|^2 \leq 2(b_k^2 q^2 + a_k^2 p^2) = 2b_k(b_k q^2 + m^2 p^2) = 8b_k V_-^k(q,p) \leq 8b_k V^k(q,p)$$

Since V^k does not increase along this part of the trajectory it follows that $V^k(q,p) \leq V^k(q_k, p_k)$. Taking (2.3), (3.4) and (3.5) into account, we obtain

$$|u|^2 \leq 8b_k V^k(q_k,p_k) \leq 8b_k V_+^k(q_k,p_k) = 4b_k W_k = U^2$$

4. AN ESTIMATE OF THE TIME OF MOTION

It follows from (2.3) and (3.13) that $a_k/8 = mU/16W_k^{1/2} \geqslant M^2\varepsilon_k/2$. Using this estimate, we extend (3.15) as follows:

$$\dot{V}^k(q,p) \leqslant -\varepsilon_k b_k q^2/4 - M^2\varepsilon_k p^2/2 \leqslant -\varepsilon_k V_+^k(q,p)/4 \leqslant -\varepsilon_k V^k(q,p)/4$$

We integrate this inequality over the half-interval $[t_k, t_{k+1})$ to get

$$t_{k+1} - t_k \leqslant \frac{4}{\varepsilon_k} \ln \frac{V^k(q_k, p_k)}{V^k(q_{k+1}, p_{k+1})} \tag{4.1}$$

Let us estimate the expression under the logarithm sign. By definition, the quadratic forms V_+^k and V_-^k are related by $V_-^k(q, p) \geqslant m^2V_+^k(q, p)/(4m^2)$. Whence, using (3.5) and the equalities $b_k = b_{k+1}/2$, $W_{k+1} = W_k/2$, we obtain

$$V^k(q_{k+1}, p_{k+1}) \geqslant V_-^k(q_{k+1}, p_{k+1}) \geqslant \frac{m^2}{4M^2} V_+^k(q_{k+1}, p_{k+1}) =$$

$$= \frac{m^2}{4M^2}\left(\frac{b_{k+1}}{2} q_{k+1}^2 + M^2 p_{k+1}^2\right) \geqslant \frac{m^2}{8M^2} V_+^{k+1}(q_{k+1}, p_{k+1}) = \frac{m^2 W_k}{32M^2}$$

It follows that (4.1) can be extended

$$t_{k+1} - t_k \leqslant \frac{4}{\varepsilon_k} \ln \frac{32M^2 V_+^k(q_k, p_k)}{m^2 W_k} = \frac{8}{\varepsilon_k} \ln \frac{4M}{m} \tag{4.2}$$

ε_k being given by (3.13). We can see that the expressions under the min sign in (3.13) are identical at $k = 0$. If (q_k, p_k) lies outside the ellipsoid with number zero, i.e. $k < 0$, then $mU/(8M^2W_k^{1/2}) > mU^2/(16\sqrt{(2)}D_1W_k^{3/2})$, and if (q_k, p_k) lies inside or on the null ellipsoid, that is, $k \geqslant 0$, then the reverse inequality holds.

We assume initially that $k < 0$. We substitute the expressions for ε_k and W_k into (4.2). This gives

$$t_{k+1} - t_k \leqslant \tau 2^{-3k/2}, \qquad \tau = \frac{32 \cdot 2^{1/4} M^3}{m\sqrt{D_1} U} \ln \frac{4M}{m} \tag{4.3}$$

The time of motion of the system from (q_k, p_k) to (q_0, p_0), i.e. from the ellipsoid with number k to that with number zero does not exceed

$$T_1 = \tau \sum_{i=k}^{-1} 2^{-3i/2} = \tau 2\sqrt{2} \frac{(2\sqrt{2})^{-k} - 1}{2\sqrt{2} - 1} \tag{4.4}$$

Now suppose that $k \geqslant 0$. In this case (4.2) takes the form

$$t_{k+1} - t_k \leqslant \tau 2^{-k/2} \tag{4.5}$$

and the time of motion from the ellipsoid with number zero to the final position (0, 0) does not exceed the sum of the series

$$T_2 = \tau \sum_{i=0}^{\infty} 2^{-i/2} = \tau \frac{\sqrt{2}}{(\sqrt{2} - 1)} \tag{4.6}$$

Up until now we have assumed that $k > k_*$ and considered a section of trajectory whose end-points lie on two neighbouring ellipsoids in the family of ellipsoids specified above. Inequalities (4.5) and (4.6) provide an estimate of the time of motion of system (1.1), (1.5) along such a segment. Now let $k = k_*$. At the point (q_*, p_*) corresponding to the initial state of the system W satisfies the inequality $W_{k^*+1} < W(q_*, p_*) \leqslant W_{k^*}$. Therefore (q_*, p_*) does not, in general, lie on the ellipsoid with number k_*. Nevertheless, at the initial instant $t = 0$ we determine a and b from (2.3) from $k = k_*$. Using an argument similar to the above it can be shown that the trajectory of system (1.1), (1.5) reaches the ellipsoid with number $k_* + 1$ and the time of motion towards this ellipsoid satisfies (4.3) if $k_* < 0$ and (4.5) if $k_* \geqslant 0$. The total time T_* of motion of the system from (q_*, p_*) to the final position (0, 0) satisfies the inequality $T_* \leqslant T_1 + T_2$, where T_1 and T_2 can be computed from (4.4) and (4.6) from $k = k_*$.

5. EXTERNAL PERTURBATIONS

We now consider the restrictions imposed on the external perturbations S. It can be seen that for $k \geqslant 0$, i.e. inside the ellipsoid with number zero, condition (3.14) is equivalent to

$$S_0 \leqslant \sigma, \quad \sigma = \frac{mU}{16\sqrt{10}M} \tag{5.1}$$

and outside the null ellipsoid (3.14), i.e. for $k < 0$, it is equivalent to

$$S_0 \leqslant \sigma 2^k \tag{5.2}$$

The least value of k along the trajectory starting at (q_*, p_*) is equal to k_*. Thus, if the point (q_*, p_*) lies inside or on the null ellipsoid, then $k_* \geqslant 0$, and (5.1) is a sufficient condition for the system under consideration to be taken from this point to the origin of the system of coordinates in a finite time using the above control law. But if (q_*, p_*) lies outside the null ellipsoid and $k_* < 0$, then such a sufficient condition is provided by inequality (5.2) for $k = k_*$.

The proposed sufficient conditions for the system to be taken to the origin are such that the maximum admissible intensity S_0 of external perturbations depends on the initial state of the system: the further away (q_*, p_*) is from (0, 0) the smaller should S_0 be. However, these conditions may be weakened if the control law is modified.

We will show that condition (5.1) is sufficient for the system to be taken from (q_*, p_*) to the origin. It has been observed above that any point of the form $(\bar{q}, 0)$ in the phase space of the system can be chosen as the final state. Then the family of ellipsoids on which the gains are altered turns out to be shifted by the vector q, the parameters of the ellipsoids remaining as before. We will assume initially that the velocity of the system satisfies the inequality

$$p_*^2 \leqslant \frac{U}{4\sqrt{2}D_1} \tag{5.3}$$

at the initial instant, that is, the point (q_*, p_*) lies on or inside the ellipsoid $W(q - q_*, p) = W_0$ (this is the null ellipsoid with centre moved to $(q_*, 0)$). We apply the control algorithm presented and move the system to the state $(q_*, 0)$. It follows from the above that for such a transition to be realized it suffices that condition (5.1) is satisfied. We choose a finite sequence of points $(\bar{q}_j, 0)$ such that $\bar{q}_0 = q_*$, $\bar{q}_J = 0$ and

$$|\bar{q}_j - \bar{q}_{j-1}| \leqslant \frac{M^2}{2D_1}, \quad j = 1, \ldots, J \tag{5.4}$$

We take the system from $(q_*, 0)$ to the origin of coordinates in J steps, applying the above control algorithm again each time. At the jth step $(\bar{q}_{j-1}, 0)$ corresponds to the initial state and $(\bar{q}_j, 0)$ to the final state of the system. Inequality (5.4) means that for any j, $(\bar{q}_{j-1}, 0)$ lies on or inside the null ellipsoid with centre at $(\bar{q}_j, 0)$. Consequently, it is sufficient if S_0 satisfies (5.1) for the transition from $(\bar{q}_{j-1}, 0)$ to $(\bar{q}_j, 0)$ to be realized.

Now suppose that (5.3) is not satisfied at the initial instant. We supplement the control algorithm by one more stage preceding all the others. The aim of the preliminary stage is to reduce the velocity of motion of the system to a value satisfying (5.3). In $H = \{(q, p) : p^2 > U/(4\sqrt{(2)}D_1)\}$ we put $u = -(U/|p|)p$. By the theorem on the variation of the kinetic energy of the system, conditions (5.1) and the definition of H it follows that the limits

$$T(q,p) \geq m^2 p^2 > \frac{m^2 U}{4\sqrt{2}D_1}$$

$$\dot{T} = \langle u + S, p \rangle \leq -(U - S_0)|p| \leq -\left(1 - \frac{\sigma}{U}\right)\left(\frac{U^3}{2\sqrt{2}D_1}\right)^{1/2}$$

hold in H, implying that the system will reach H in a finite time. As soon as the trajectory reaches the boundary of H the preliminary stage of control is completed and the realization of the above algorithm of the stepwise transition of the system into the final state begins.

6. RESULTS OF MODELLING

The control law proposed was used in a numerical model of the controlled motion of a two-link unit on a fixed support. The hinge angles of the links in the stationary system of coordinates were chosen as the generalized coordinates of the system. The expression for the kinetic energy of the unit has the form $T = R_1\dot{q}_1^2 + R_2\dot{q}_2^2 + 2R_3\cos(q_1 - q_2)\dot{q}_1\dot{q}_2$. Calculations were performed for the following parameter values: $R_1 = 13.9$; $R_2 = 2.1$; $R_3 = 3$ kg m^2. The eigenvalues of the inertia matrix are between $m^2 = 1.9$ and $M^2 = 14.1$ kg m^2, while the norm of the partial derivatives of the matrix is bounded by $D = 3$. The maximum admissible magnitude of the vector of control torques was chosen to be equal to $U = 500$ N m. The system was moved from the state $q_1 = 0.5$; $q_2 = 1$ rad, $\dot{q}_1 = \dot{q}_2 = 0$ rad/s to the position $q_1 = q_2 = \dot{q}_1 = \dot{q}_2 = 0$.

The value of W at the point corresponding to the initial state of the system is equal to $W(0) = 395$ and the quantity determining the null ellipsoid is equal to $W_0 = 1171$. Since $W_0/4 < W(0) < W_0/2$, the first value of k is equal to one. The initial point of the trajectory lies inside the null ellipsoid, so S_0 only needs to satisfy condition (5.1), which in the case in question takes the form $S_0 \leq 3.6$. The perturbing moments in the model are given by the constant vector-valued function $S(t) = (0; 3.6)$.

In Figs 1 and 2 we show the graphs of the time dependence of the phase coordinates of the system The dashed curve corresponds to the generalized coordinates (rad) and the dotted curve corresponds

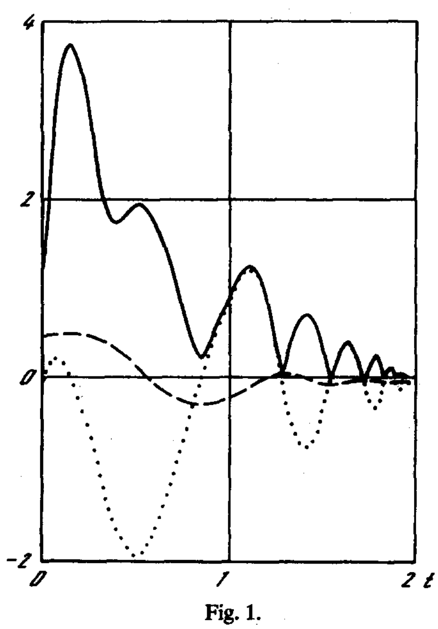

Fig. 1.

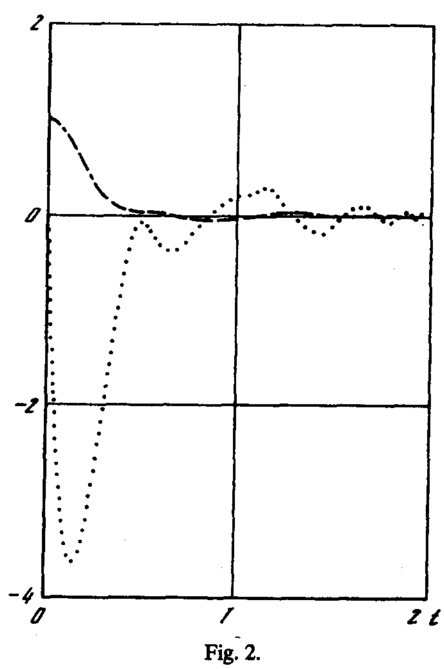

Fig. 2.

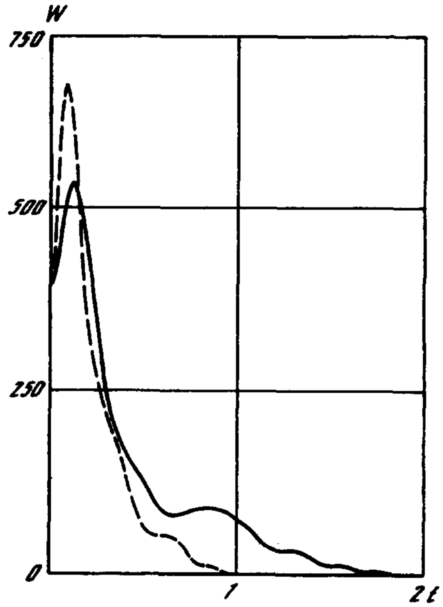

Fig. 3.

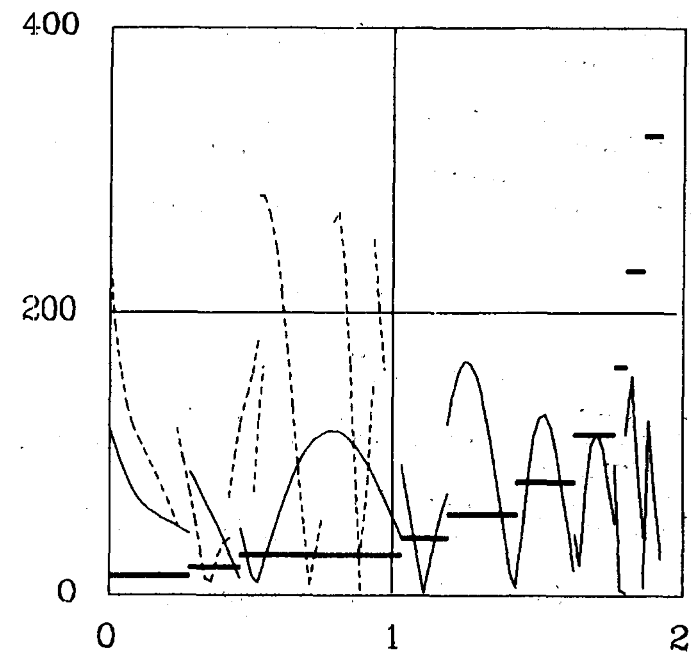

Fig. 4.

to the velocities (rad/s). Figure 1 describes the motion of the first link and Fig. 2 the motion of the second link. The solid line in Fig. 1 shows the time dependence of the Euclidean distance in the phase space $q, q^{\cdot}$ between the actual state of the system and the terminal state. The integration of equations was terminated when the distance became less than 0.01. During the time of integration of the equations the feedback factors in (1.5) changed 15 times. The solid curve in Fig. 3 illustrates the behaviour of W along the trajectory. It can be seen that neither the distance between the actual and the final states of the system nor the function W depend monotonically on time.

In Fig. 4 we show the time dependence of the absolute value of the vector of control torques (the thin curve) and the magnitude of the gain a_k (the step function). In accordance with the algorithm, the feedback factors a_k and b_k are chosen in such a way that for any admissible values of the unknown parameters, that is, the elements of the inertia matrix and the components of the vector of perturbing torques, the control constraints (1.3) are satisfied along the resulting trajectory of motion. For a specified mechanical system the domain of variation of these parameters contracts significantly and the choice of the gains may turn out to be unnecessarily rough. One can see that in the case under consideration the control torques obtained are much smaller than the maximum allowed magnitude of U. This is why we modelled the motion of a two-link unit controlled by the same laws, but with gains a_k and b_k twice as large as those prescribed by the algorithm. In Fig. 3 the dashed curve represents the time dependence of W, while in Fig. 4 it represents the absolute magnitude of the vector of control torques for this control method. The time it takes the system to reach the final position has been reduced by a factor of two, while the control satisfies (1.3) as before, with a substantial margin.

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